PLANE MOTION OF RIGID BODIES

7. Topic: **Forces & accelerations**

Engage:

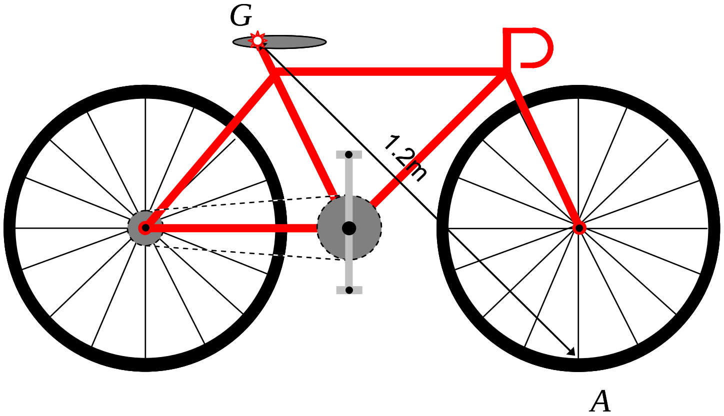
Ride your bike into class, apply the front brakes too hard so that the rear wheel just begins to lift off the ground. Wear a helmet!

Explore:

Dismount from your bike and standing on the floor, demonstrate how applying the front brakes when gripping the handle-bars causes the rear wheel of the bike to lift off the floor. Use a mountaineering karabiner or other similar clip to suspend a backpack of books on the bike saddle. Invite members of the class, standing on the floor, to repeat the demonstration with different numbers of books in the backpack and with backpack suspended at different locations. Talk about the moment of inertia as a measure of the resistance of a body (the bike and backpack in this case) to angular acceleration (tendency for the rear wheel to lift in this case) in the same way that mass is a measure of body's resistance to acceleration (i.e. Newton's Second Law).

Explain:

When the front brake is locked so that there is no motion of the front wheel relative to the bike, then the bike (including the front wheel) and backpack rotate about the point on the front wheel that is in contact with the road, i.e. *A*



The moment of inertia of a body of mass, *m* about an axis is: $I = \int_m r^2 dm$

where r is the perpendicular distance from the axis to an arbitrary element dm. The moment of inertia for a solid cuboid of height h, width, w depth d and mass m is:

$$I_h = \frac{m}{12}\left(w^2 + d^2\right)$$

where I_h is the moment of inertia about the axis through the center of mass parallel to the height dimension. So, for a pile of textbooks in a backpack of mass 3.8 kg and of dimensions 0.12×0.21×0.26m:

$$I_h = \frac{m}{12}\left(w^2 + d^2\right) = \frac{3.8}{12}\left(0.21^2 + 0.26^2\right) = 0.035\,\text{kg.m}^2$$

When the moment of inertia, I_G about the body's center of mass is known then the moment of inertia, I about any other parallel axis is given by the parallel axis theorem:

$$I = I_G + mb^2$$

where b is the perpendicular distance between the axes. So if we assume that the center of mass of the backpack is at the location, G in the diagram above then, its moment of inertia about A, I_A is given by:

$$I_A = I_G + mb^2 = 0.035 + (3.8 \times 1.2^2) = 0.035 + 5.47 = 5.51\,\text{kg.m}^2$$

We neglected the moment of inertia of the backpack but this is inconsequential because the second term in the expression above dominates and location of the backpack and the mass of the books has a huge effect on the resistance to rotational acceleration.

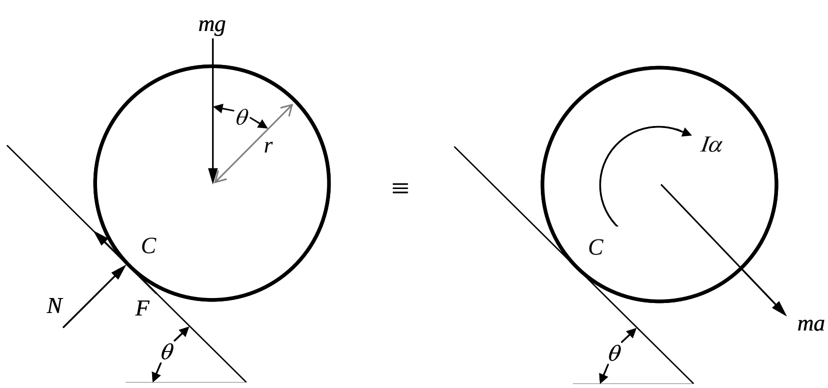
Elaborate:

The equation of rotational motion is given by: $\sum M_G = I_G\alpha$

where the moments M_G are considered about an axis through the center of mass and α is the angular acceleration of the body.

When a child rolls down a steep grassy bank (roly-poly) we can calculate their acceleration by using the above expression and a few simplifying assumptions. Let us assume that the child has uniform radius of 0.15m, a mass of 40kg and a moment of inertia of 0.8kg.m^2; that the grassy bank has a 45 degree slope (θ =45°) and that we can represent the child by a cylinder or imagine them in a plastic pipe.

<u>Then, drawing a free body diagram</u>:



Considering motion about C, the equation of rotational motion is:

$$\sum M_c = mgr\sin\theta = (ma)r + I\alpha$$

Now, the linear acceleration is, $a = r\alpha$ and the radius of gyration, k is defined by $I = mk^2$,
so $mgr\sin\theta = (mr\alpha)r + mk^2\alpha$

rearranging $\alpha = \dfrac{r\ g\sin\theta}{r^2 + k^2}$

and $a = \dfrac{r^2 g\sin\theta}{r^2 + k^2} = \dfrac{0.15^2 \times 9.81 \times \sin 45}{0.15^2 + (0.8/40)} = 3.67\ \text{ms}^{-2}$

If the child starts from rest and the bank is 5m high then their velocity at the bottom can be calculated from:

$$v^2 - u^2 = 2as$$

i.e. $v = \sqrt{2as} = \sqrt{2 \times 3.67 \times 5} = 6.1\,\text{m/s}$

so the child will have a speed of 6.1 m/s (≡13.6 mph) at the bottom of the grassy bank.

Evaluate:

Ask students to attempt the following example:

Example 7.1

A pizza cutter is pushed through a thick crust pizza using a handle held such that the driving force is at 45 degrees to the cutting board. The cutting wheel and handle have a total mass of 500g. The blade is of radius 50mm with an average thickness 0.5mm and the cylindrical handle is 180mm long with a radius of 15mm. The coefficient of friction between the pizza and the blade is 0.01 and it can be assumed that the pizza exactly opposes the magnitude of the driving force with a force acting towards the axis of the cutting wheel at 10mm above the cutting board. Calculate the constant driving force required to cut through a pizza 30cm wide exiting at a velocity of 1.5m/s from a standing start – is this viable for a 12 year old child?

Solution:

Volume of pizza cutter wheel,

$$V = \pi r^2 t = \pi 0.05^2 \times 0.0005 = 3.93 \times 10^{-6}\ \text{m}^3$$

So for stainless steel of density, $\rho = 8000\ \text{kg.m}^{-3}$,

the mass of the pizza cutter wheel is

$$m = \rho V = 8000 \times 3.93 \times 10^{-6} = 0.031\,\text{kg}$$

and the moment of inertia of the wheel about its axis of rotation is given by:

$$I = \frac{mr^2}{2} = \frac{0.031\times 0.05^2}{2} = 3.93\times 10^{-5}\,\text{kg.m}^2$$

The mass of the handle is 0.469kg (= 0.5-0.031) and its moment of inertia about an axis parallel to the axis of rotation of the wheel and though its center of mass is

$$I = \frac{m}{12}\left(3r^2 + h^2\right) = \frac{0.469}{12}\left(\left(3\times 0.015^2\right) + 0.18^2\right) = 0.00129\,\text{kg.m}^2$$

And using the parallel axis theorem to shift this to be about the center of rotation of the cutter wheel:

$$I = I_G + mb^2 = (0.00129) + \left(0.469\times 0.05^2\right) = 0.0025\ \text{kg.m}^2$$

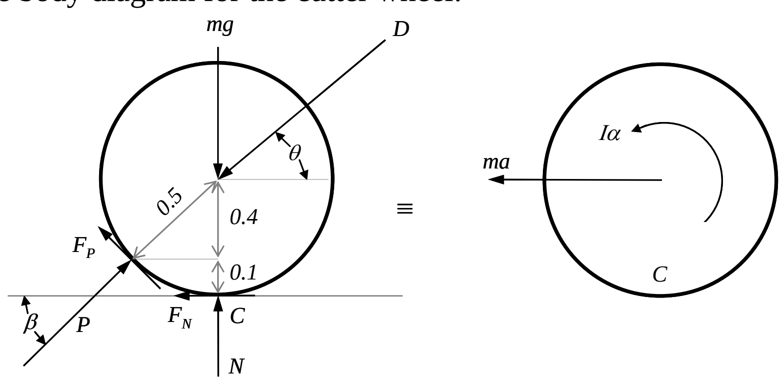
so the moment of inertia of the wheel is negligible compared to the heavy handle.

For an exit velocity of 1.5 m/s after crossing a pizza of diameter 0.3m and assuming an entry velocity of zero, using:

"$v^2 - u^2 = 2as$" we have $\quad a = \dfrac{v^2 - u^2}{2s} = \dfrac{1.5^2}{0.6} = 3.75\,\text{ms}^{-2}$

and for the cutter wheel $\quad a = r\alpha$ so $\quad \alpha = \dfrac{a}{r} = \dfrac{3.75}{0.05} = 75\,\text{rad/s}^2$

Drawing a free body diagram for the cutter wheel:



For motion about C, consider the equation of rotational motion:

$$\sum M_c = 0.5D\cos\theta - 0.1P\cos\beta - 0.3P\sin\beta + 0.1F_p\sin\beta - 0.3F_p\cos\beta = (ma)r + I\alpha = m\alpha r^2 + I\alpha$$

now, $D = P = (F_p/\mu)$ so re-arranging we can obtain:

$$D = \frac{\alpha\left(mr^2 + I\right)}{0.5\cos\theta - (0.1\cos\beta + 0.3\sin\beta) + \mu(0.1\sin\beta - 0.3\cos\beta)}$$

and substituting values for the parameters, to obtain:

$$D = \frac{75\times\left(\left(0.5\times 0.05^2\right) + 0.0025\right)}{\frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \left(\frac{4}{50} + \frac{9}{50}\right) + 0.01\left(\frac{3}{50} - \frac{12}{50}\right)} = \frac{0.281}{0.09} = 3.1\text{N}$$

So, the driving force required is 3.1N which is very small.